Role of Bell Singlet State in the Suppression of Disentanglement

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The stability of entanglement of two atoms in a cavity is analyzed in this work. By studying the general Werner states we clarify the role of Bell-singlet state in the problem of suppression of disentanglement due to spontaneous emission. It is also shown explicitly that the final amount of entanglement depends on the initial ingredients of the Bell-singlet state.

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One of the specific features of quantum world is the existence of quantum coherence which forms the basis of describing wide varieties of phenomena including superconductivity and Bose-Einstein condensation of cold atoms. During the last decade, another aspect of quantum coherence, namely, quantum entanglement[1], has been recognized as the essential element of quantum computing[2]. In order to realize quantum information processing, stability of entanglement of quantum subsystems is one of the important problems that requires careful analysis. Instabilities of quantum entanglement can be generated through different mechanisms[3]. In general, an entangled state of a closed system can be disentangled by its own dynamics[4]. On the other hand, due to decoherence, system and environment interaction might not preserve initially entangled state. However, decoherence can also be a dynamical effect if one includes the quantum fluctuation of vacuum. In fact, such fluctuation is the origin of spontaneous emission which can reduce entangled state to separable state via photon emission.

The recent work of Yu and Eberly[5] has discussed the finite-time disentanglement via spontaneous emission. In their system two non-interacting atoms are coupled to two separate cavities(environments). As a result, the dynamical evolution of the atoms are independent and, depending on initial state, the effect of spontaneous emission can drive the system to disentangled in finite time. However it is not clear if the disentangle phenomenon will persist if the atoms are allowed to interact. Intuitively, it is easy to imagine that for two atoms interacting in a lossless cavity, the photon emits by one atom during spontaneous emission can be absorbed by the other. As a result, entanglement might be preserved through this mechanism. In fact the above photon process is equivalent to the interaction between atoms by exchanging photon. Furthermore it is also more practical for constructing the quantum circuit inside one cavity instead of distribute the atoms in different separate ones. Consequently, it is inevitable to include the effects of interaction among atoms for any discussions on disentanglement via spontaneous emission. This problem has also been addressed in the interesting work by Tanaś and Ficek[6]. By putting two atoms inside the same cavity they showed that the entanglement exhibits oscillatory behavior, and the amount of entanglement is directly related to the population of the slowly decaying Bell-singlet state in the long time limit. Their results is interesting since it indicates that the Bell-singlet state is stable against photon emissions. To justify this point more concretely, it is necessary to explore further on the role of Bell-singlet in the suppression of disentanglement. This is what will be discussed in this work.

Our model system is the same as the one employed in [6] except for the fact that we neglect the spatial dependent of the atoms to avoid complication. Such simplification will make the role of spin singlet state more prominent in the suppression mechanism. The model consists of two two-level atoms A and B inside a lossless cavity. These atoms are considered as identical and allowed to interact by exchanging photon inside the cavity which is viewed as the environment. The coupling between the system and the environment is the origin of the disentanglement. The Hamiltonian of the total system is given by $H_T = H_s + H_{sb} + H_b$. H_s, H_b and H_{sb} are atomic, the bath and atoms-bath interaction Hamiltonian respectively($\hbar = 1$):

$$H_s = \frac{1}{2}\omega_0 \Sigma_z \tag{1}$$

$$H_b = \sum_k \omega_k (a_k^\dagger a_k + \frac{1}{2}) \tag{2}$$

$$H_{sb} = \sum_k (g_k^* \Sigma_- a_k^\dagger + g_k \Sigma_+ a_k) \tag{3}$$

where $\Sigma_i \equiv \sigma_i^A + \sigma_i^B$, i can either be $\{x, y, z\}$ or $\{+, -\}$ for raising and lowering operations and $a_k(a_k^\dagger)$ is the photon annihilation(creation) operator of mode k. Here we have assumed that these atoms are identical such that $\omega_A = \omega_B \equiv \omega_0$ and they couple to photon mode k with the same strength g_k. In this work we assume that the atoms are entangled but not with the bath at $t = 0$. Furthermore, the initial bath state is assumed to be the vacuum state. The initial total state is then given by the following product state,

$$|\psi_{total}\rangle = |\psi\rangle_{AB} \otimes |0\rangle. \tag{4}$$

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Here $|\psi\rangle_{AB}$ is the entangled initial state of the atoms and $|0\rangle$ denotes the vacuum state of the cavity. The master equation of atoms in the Schrödinger picture can be obtained as follows:

$$\begin{aligned}\dot{\rho}_t &= -i[H_s,\rho_t] - \{\Sigma_+ f(t)\Sigma_-\rho_t - f(t)\Sigma_-\rho_t\Sigma_+ \\ &\quad +\rho_t f^\dagger(t)\Sigma_+\Sigma_- - f^\dagger(t)\Sigma_+\rho_t\Sigma_-\}. \end{aligned} \tag{5}$$

Here $f(t) \equiv \sum_k \int_0^t dt' C_k(t-t')e^{i\omega_0(t-t')}$ with $C_k(t-t')$ given by the photon correlation function

$$C_k(t-t') = Tr(\tilde{A}_k(t)\tilde{A}_k^\dagger(t')|0\rangle\langle 0|) \tag{6}$$

where $\tilde{A}_k(t) \equiv \sum_k g_k\tilde{a}_k(t)$ and $\tilde{a}_k(t)$ is the photon annihilation operator in interaction picture. With $f(t) = f_R(t)+if_I(t)$, one can further arrange Eq.(5) into unitary and decoherent evolutions as follows:

$$\begin{aligned}\dot{\rho}_t &= -i[H_s + f_I(t)\Sigma_+\Sigma_-,\rho_t] \\ &\quad -f_R(t)\{[\Sigma_+,\Sigma_-\rho_t] + [\rho_t\Sigma_+,\Sigma_-]\}. \end{aligned} \tag{7}$$

The physical meaning of Eq.(7) can be understand by expanding the commutators. Inside the first commutator, the hamiltonian which contributes to the unitary evolution contains the energy eigenvalues to the second order corrections and dipole interactions between A and B atoms. One can see that the coupling of dipole interaction is identical to the energy correction. This is due to the fact that both atoms couple to photon with the same strength. The rest of Eq.(7) which is non-unitary could intuitively imply decoherent evolution. However it turns out that in our case this intuitive picture is illusive. Apart from the spontaneous real photon emission process which definitely gives rise to decoherence, the non-unitary dynamics also includes more non-local photon-exchange interactions which turns out to be the main driving force for the suppression of disentanglement mentioned earlier. Explicitly, the master equation is:

$$\begin{aligned}\dot{\rho}_t &= -i[\frac{1}{2}(\omega_0 + f_I(t))\Sigma_z + f_I(t)(\sigma_+^A\sigma_-^B + \sigma_+^B\sigma_-^A),\rho_t] \\ &\quad -f_R(t)\{\sigma_+^A\sigma_-^B\rho_t + \rho_t\sigma_+^A\sigma_-^B + \sigma_+^B\sigma_-^A\rho_t + \rho_t\sigma_+^B\sigma_-^A \\ &\qquad -2\sigma_-^A\rho_t\sigma_+^B - 2\sigma_-^B\rho_t\sigma_+^A\} \\ &\quad -2f_R(t)\{\sigma_+^A\sigma_-^A\rho_t + \rho_t\sigma_+^A\sigma_-^A - \sigma_-^A\rho_t\sigma_+^A - \sigma_-^A\rho_t\sigma_+^A\} \\ &\quad -2f_R(t)\{\sigma_+^B\sigma_-^B\rho_t + \rho_t\sigma_+^B\sigma_-^B - \sigma_-^B\rho_t\sigma_+^B - \sigma_-^B\rho_t\sigma_+^B\}.\end{aligned}$$

The general solution for the master equation can be found by constructing the Kraus operator $K_\mu(t)$ which gives the density matrix $\rho(t)$ in terms of the initial state $\rho(0)$ as

$$\rho(t) = \sum_\mu K_\mu(t)\rho(0)K_\mu^\dagger(t), \tag{8}$$

where the Kraus operators $K_\mu(t)$ satisfy $\sum_\mu K_\mu(t)K_\mu^\dagger(t) = I$ for all t. The advantage of using the Kraus representation is the fact that $\rho(t)$ satisfies all the requirements of the density matrix which are positivity and $Tr\rho = 1$. However, in this work it is more easier to obtain $\rho(t)$ by solving the coupled differential equation directly. Due to the non-unitary evolution of the master equation, such $\rho(t)$ does not a priori satisfy the above requirements of density matrix and therefore it requires extra care to ensure the results are satisfactory. Accordingly, we have checked in all calculations that our solution are indeed a good density matrix for all time.

It is well-known that a good definition of entangled mixed state is still lacking for general system. However, for 2×2 system it has been shown by Peres[7] and Horedicki's[8] that Positive Partial Transpose (PPT) of the density matrix is a good criterion for characterizing the separability of states. It will be shown latter that both definitions are equivalent. Furthermore, the distillation protocols first invented by Bennett et. al.[9] are performed by LOCC and can convert a large number of mixed state with ingredient of entanglement into a smaller number of maximally entangled pure state which is the most important resource in quantum information processing (QIP). There are many efforts on figuring out the relation between distillability and separability of mixed state density matrix[10]. For 2×2 system, it has been shown that the mixed state can be distilled if it violates PPT criterion[10]. In this work, we will shown latter that the state of two coupled atoms in cavity undergoing the dynamical bath interaction become inseparable, stable and distillable.

In what follows we will choose the initial state with the form in computational basis of $\{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}$ where $|-\rangle(|+\rangle)$ represents the ground(excited) state of the atom:

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{9}$$

these states contains a special subclass of mixed state, the general Werner states W_F[11**?**] parameterized by a single real parameter F is: $\rho_{11} = \rho_{44} = \frac{1}{3}(1-F)$, $\rho_{22} = \rho_{33} = \frac{1}{6}(1+2F)$ and $\rho_{23} = \rho_{32} = \frac{1}{6}(1-4F)$. One can easily check that the conditions of density state require that $0 \le F \le 1$ and W_F is inseparable for $F > \frac{1}{2}$. F which can be regards as the fidelity of W_F , $\langle\Psi^-|W_F|\Psi^-\rangle$, relative to singlet state also quantifies the upper bound of the distillable maximally entangled singlet states by LOCC. We note that Wootter's concurrence[12] is given by $C(\rho) = Max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ where λ's are the square root of the eigenvalues of spin flow matrix defined by ρ: $R = \rho(\sigma_y^A\otimes\sigma_y^B)\rho^*(\sigma_y^A\otimes\sigma_y^B)$, subtracting in decreasing order. A state contains no entanglement with $C = 0$, while maximally entangled with $C = 1$. Hence, the concurrence of ρ is given by

$$C(\rho) = 2Max\{0, |\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}\}. \tag{10}$$

As to the PPT criterion,

$$\rho_{11}(t)\rho_{44}(t) \ge |\rho_{23}(t)|^2, \tag{11}$$

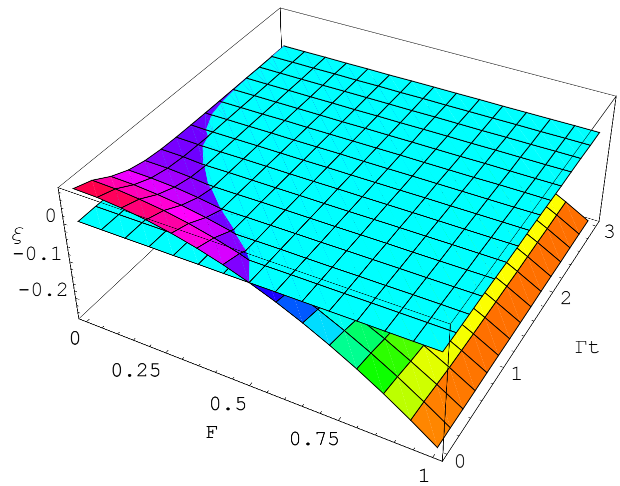

FIG. 1: (Color online) The $\xi(t,F)$ plot which is PPT criterion with $0 \leq F \leq 1$. For $F > 1/2$, ξ is negative for all time, which indicates that the inseparability is preserved under evolution. For $F \leq 1/2$, initially the ρ_F is separable due to positive of ξ, while within finite time, they become inseparable.

which gives the same consequence of concurrence in Eq. (10). For convenience of following discussion, we define $\xi(t) \equiv \rho_{11}(t)\rho_{44}(t) - |\rho_{23}(t)|^2$ which is positive for separability, whereas negative for entangled state.

Due to the special form of the initial states considered in this work, we will give only the relevant matrix elements explicitly. In the Markovian limit such that $\int_0^t dt' f_R(t') \equiv \frac{\Gamma}{2}t$ and $\int_0^t dt' f_I(t') \equiv \frac{\gamma}{2}t$,these matrix elements are:

$$\rho_{11}(t) = \rho_{11}(0)e^{-2\Gamma t} \quad (12a)$$

$$\rho_{22}(t) = \frac{S_- + e^{-2\Gamma t}S(t)}{4} - e^{-\Gamma t}\rho_I(0)\sin\gamma t \quad (12b)$$

$$\rho_{33}(t) = \frac{S_- + e^{-2\Gamma t}S(t)}{4} + e^{-\Gamma t}\rho_I(0)\sin\gamma t \quad (12c)$$

$$\rho_{23}(t) = \frac{-S_- + e^{-2\Gamma t}S(t)}{4} + ie^{-\Gamma t}\rho_I(0)\cos\gamma t \quad (12d)$$

$$\rho_{32}(t) = \frac{-S_- + e^{-2\Gamma t}S(t)}{4} - ie^{-\Gamma t}\rho_I(0)\cos\gamma t \quad (12e)$$

$$\rho_{44}(t) = 1 - \rho_{11}(t) - \rho_{22}(t) - \rho_{33}(t). \quad (12f)$$

Here, $S(t) \equiv S_+ + 4\rho_{11}(0)\Gamma t$ and $S_- \equiv \rho_{22}(0) + \rho_{33}(0) - \rho_{23}(0) - \rho_{32}(0)$, $S_+ \equiv \rho_{22}(0) + \rho_{33}(0) + \rho_{23}(0) + \rho_{32}(0)$. We also denote $\rho_I(0)$ as the imaginary part of $\rho_{23}(0)$. It is obvious that the exponential damping factors in these equations are due to non-unitary evolution.

Now we are in the position to show the suppression of disentanglement in this system. Taking the general Werner states as initial state, $\rho(0) = W_F \equiv \rho^F(0)$, the evolution is then given by

$$\rho_{11}^F(t) = \frac{(1-F)}{3}e^{-2\Gamma t} \quad (13a)$$

$$\rho_{22}^F(t) = \frac{F}{2} + e^{-2\Gamma t}\{\frac{(1-F)}{6} + \frac{(1-F)}{3}\Gamma t\} \quad (13b)$$

$$\rho_{23}^F(t) = -\frac{F}{2} + e^{-2\Gamma t}\{\frac{(1-F)}{6} + \frac{(1-F)}{3}\Gamma t\} \quad (13c)$$

$$\rho_{44}^F(t) = 1 - F - \frac{2(1-F)(1+\Gamma t)}{3}e^{-2\Gamma t}. \quad (13d)$$

Here, $\rho_{33}^F(t) = \rho_{22}^F(t)$ and $\rho_{32}^F(t) = \rho_{23}^F(t)$. Note that the evolution effect comes solely from the non-unitary evolution. The numerical result of $\xi(t,F)$ is shown in Fig.(1). For any initially entangled Werner states with $F > 1/2$, one can see from Fig.(1) that $\xi(t,F) < 0$ and as a result the inseparable state remains entangled for all time. Hence, there is **NO** disentanglement effect at all even with spontaneous emission. Moreover the entanglement can be enhanced and attains stable state in finite time. To be concrete, we present the results of enhanced entanglement in Fig.(2a) where we plot the time dependence of concurrences for two initial states with $F = 0.75$ and $F = 0.25$. Our result clearly shows that, for $F = 0.75$, the degree of entanglement is enhanced and reaches a saturated value in finite time. For the case of $F = 0.25$ it is noted that $W_{0.25}$ is just an equal mixing of all possible eigenstates of the system: $W_{0.25} = \frac{1}{4}\mathbf{I}$. Accordingly, $W_{0.25}$ is a classically correlated state with no entanglement. However our result in Fig.(2a) has shown that entanglement is generated in finite time. This last result of generating entangled state from no-entangled state naturally raises an interesting question, namely, will it be true that entanglement can always be stabilized or even enhanced with the presence of non-unitary evolution? Unfortunately, the answer is negative! To see this, let us consider the following entangled initial state $\tilde{\rho}$ with a single parameter $0 \leq a \leq 1$: $\tilde{\rho}_{11} = \frac{1}{3}a$, $\tilde{\rho}_{44} = \frac{1}{3}(1-a)$ and $\tilde{\rho}_{22} = \tilde{\rho}_{33} = \tilde{\rho}_{23} = \tilde{\rho}_{32} = \frac{1}{3}$. Note that $\tilde{\rho}$ is entangled for all a initially($C(0) \neq 0$). However, for instance, entanglement of the initial state with $a = 0$ is decaying asymptotically, while for $a = 0.5$, entanglement is vanishing abruptly(See Fig.(2b)). This might sound puzzling and seem contradicting to the result of the classically correlated state, namely $W_{0.25}$. However, this contradiction is illusive. To resolve this puzzle, first we note that the Bell-singlet state has been shown being much more stable than the other Bell states under dynamical evolution[4]. Secondly, the initial state $\tilde{\rho}$ is a mixture of Bell states without the Bell-singlet. Thus it seems that the instability of the non-singlet Bell states is probably the reason for disentanglement to happen. If this is true then the existence of singlet state is the essential ingredient for enhanced entanglement to happen. To certify this point, one can add a small amount of singlet states to $\tilde{\rho}$. Then

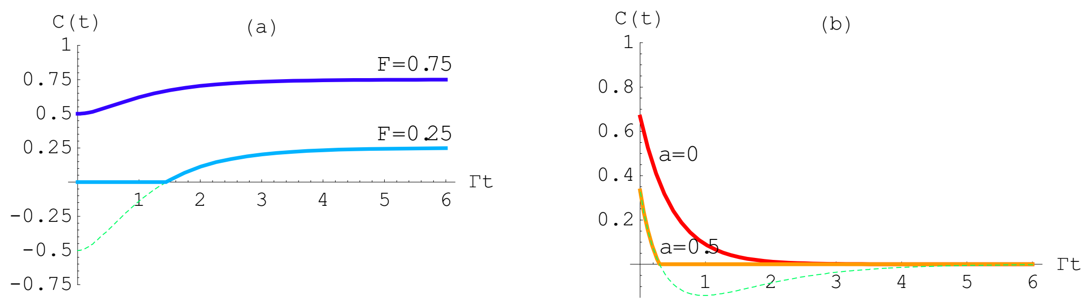

FIG. 2: (Color online)(a) Concurrence of Werner state for the cases of $F = 0.75$ and $F = 0.25$. One can see the entanglement is enhanced and become stable for both case within finite time. (b) Concurrence of $\tilde{\rho}(t)$ for the cases of $a = 0.5$ and $a = 0$. Disentanglement can always happen suddenly or asymptotically.

the modified $\tilde{\rho}^{\epsilon}$ can be expressed as:

$$\begin{aligned}\tilde{\rho}^{\epsilon} &= \frac{1}{3}\{2\epsilon|\Psi^-\rangle\langle\Psi^-| + a|++\rangle\langle++| + 2|\Psi^+\rangle\langle\Psi^+| \\ &+(1-a-2\epsilon)|--\rangle\langle--|\},\end{aligned} \tag{14}$$

where $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle)$ and ϵ is an infinitesimal parameter. In the long time limit, the concurrence of $\tilde{\rho}^{\epsilon}(\infty)$ is $\frac{2\epsilon}{3}$ which is just the amount of singlet states in $\tilde{\rho}^{\epsilon}$. Therefore, the singlet part of $\tilde{\rho}^{\epsilon}$ is being preserved under evolution, whereas the triplet states will be affected and dissipated by decoherence. In fact, the same evidence can also be further generalized to the initial states given by Eq.(9). Indeed, by taking long time limit of $\rho(\infty)$, one obtain

$$\rho(\infty) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{S_-}{4} & -\frac{S_-}{4} & 0 \\ 0 & -\frac{S_-}{4} & \frac{S_-}{4} & 0 \\ 0 & 0 & 0 & 1-\frac{S_-}{2} \end{pmatrix} \tag{15}$$

where S_- is defined earlier. The corresponding concurrence is $C(\rho(\infty)) = \frac{S_-}{2}$. The condition with entanglement implies $S_- \neq 0$. This result shows that, once the initial state $\rho(0)$ contains some ingredients of singlet even with $\rho(0)$ being no entangled state, within finite time, the state attains stabilized entanglement. For ρ^F, $C(\rho^F(\infty)) = F$ with F being the fraction of singlet state. Therefore, if the amount of singlet state is nonzero, namely $F \neq 0$, then the concurrence is also nonzero within finite time and hence the state becomes entangled. In contrast to the stable Bell singlet, the triplet state is unstable due to spontaneous emission. This instability is just a consequence of Dicke model, namely the triplet state can decay to the ground state whereas the singlet state cannot. The reason for the singlet state being stable is due to the vanishing of the total dipole and, in the long wavelength limit considered here, the state decouples from the photon bath. One should notice that, without real photon exchange between atoms such as the case considered by Yu and Eberly[5] where two atoms are put in two separate cavities, disentanglement always happens whatever the initial state is.

To summarize, we have shown that the interactions induced by the vacuum fluctuations provide both dipole interactions and nonlocal photon exchanged interactions. The decoherent effect of spontaneous emission is shown being suppressed by the nonlocal photon exchanged interactions coming from the non-unitary dynamics. In passing we would like to stress again that such suppression mechanism is due to the real photon exchange process. With the success of suppressing decoherence, it is further shown that entanglement can be stabilized and even enhanced. Surprisingly, the enhancement process can be saturated in finite time. The basic condition for stabilization and enhancement of entangled state is the existence of singlet state in any mixed state and supporting examples are also provided in this work.

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